Calculus II - Day 7

Prof. Chris Coscia, Fall 2024
Notes by Daniel Siegel

25 September 2024

The Ratio and Root Tests

Goals for today:

- Use the Ratio and Root tests to determine whether series converge absolutely or diverge.
- Practice deciding which test to use for a given series.

Recall: Absolute versus conditional convergence:

$\sum a_k$

- Converges absolutely if $\sum |a_k|$ converges (and therefore $\sum a_k$ converges, too!)
- Converges conditionally if $\sum a_k$ converges, but $\sum |a_k|$ diverges.

Ex.

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

Let's show this series converges absolutely, which means it converges:

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right|$$

Since $|\sin(k)| \leq 1$ for all k, we have:

$$\left| \frac{\sin(k)}{k^2} \right| \leq \frac{1}{k^2}$$

We know $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so:

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2} \text{ converges absolutely}$$

$\Rightarrow$ it $\boxed{\text{converges}}$.

Geometric series:

$$\sum_{k=0}^{\infty} ar^k \text{ converges to } \frac{a}{1-r} \text{ if } |r| < 1$$

$$\text{diverges if } |r| \geq 1$$

For a convergent geometric series $\left(\sum a_k = \sum ar^k\right)$:

1. $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{ar^{k+1}}{ar^k}\right| = |r| < 1$
2. $|a_k|^{1/k} = \left|ar^k\right|^{1/k} = |a|^{1/k} \cdot \left|r^k\right|^{1/k} = |a|^{1/k} \cdot |r|$

   As $k \to \infty$, this approaches $|r| < 1$.

Statement: The ratio between consecutive terms is $|r| < 1$ always, and the Kth root of the Kth term approaches $|r| < 1$ in the limit. This idea works more broadly!

The Ratio Test

Let $\sum a_k$ be an infinite series and let

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

1. If $r < 1$, the series converges absolutely.
2. If $r > 1$ or ~~r does not exist (DNE)~~, the series diverges.
3. If $r = 1$, the test is inconclusive (anything can happen)

Note: Professor crossed out 'does not exist' in (2) due to a student's counterexample.

The Root Test

Let $\sum a_k$ be an infinite series and let

$$\rho = \lim_{k\to\infty} (a_k)^{1/k}$$

Same conclusions as the Ratio Test:

- If $\rho < 1$, the series converges absolutely.
- If $\rho > 1$, the series diverges.
- If $r = 1$, the test is inconclusive (anything can happen)

Usually, one of these tests is easier than the other.

Ratio test example

Determine whether $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k^2+4)}{e^k}$ converges absolutely.

$$a_k = \frac{(-1)^{k+1}(k^2+4)}{e^k} \quad \rightarrow \quad a_{k+1} = \frac{(-1)^{k+2}((k+1)^2+4)}{e^{k+1}}$$

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+2}((k+1)^2+4)}{e^{k+1}} \cdot \frac{e^k}{(-1)^{k+1}(k^2+4)}\right|$$

$$= \lim_{k\to\infty} \left|\frac{(-1)(k^2+2k+5)}{e(k^2+4)}\right|$$

$$= \frac{1}{e} \lim_{k\to\infty} \frac{k^2+2k+5}{k^2+4} = \frac{1}{e}$$

Since $\frac{1}{e} < 1$, the series converges absolutely.

<u>Ex.</u> $\sum_{k=1}^{\infty} \frac{10^k}{k!}$

$$a_k = \frac{10^k}{k!} \quad \rightarrow \quad a_{k+1} = \frac{10^{k+1}}{(k+1)!}$$

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k}\right|$$

$$= \lim_{k\to\infty} \frac{10 \cdot k!}{(k+1)!} = \lim_{k\to\infty} \frac{10}{k+1}$$

$$k! = k \cdot (k-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \quad \text{and} \quad (k+1)! = (k+1) \cdot k \cdot (k-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

(crossing out like terms):

$$\frac{k!}{(k+1)!} = \frac{k \cdot (k-1) \cdot \cdots \cdot 1}{(k+1) \cdot k \cdot (k-1) \cdot \cdots \cdot 1} = \frac{1}{k+1}$$

$$\lim_{k\to\infty} \frac{10}{k+1} = 0 < 1$$

$\Rightarrow$ The series converges.

<u>Ex.</u> $\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{k!}$

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{(-1)^k k^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{k!}{(k+1)!} \cdot \frac{(k+1)^{k+1}}{k^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{1 \cdot (k+1)^{k+1}}{(k+1) \cdot k^k}\right|$$

$$= \lim_{k\to\infty} \left| \frac{(k+1)^k}{k^k} \right| = \lim_{k\to\infty} \left(\frac{k+1}{k} \right)^k = e > 1$$

$\Rightarrow$ The series $\boxed{\text{diverges}}$.

Back to the Root Test:
Let $\rho = \lim_{k\to\infty} |a_k|^{1/k}$.

- The series converges absolutely if $\rho < 1$.
- The series diverges if $\rho > 1$.

<u>Ex.</u> $\sum_{k=1}^{\infty} \left(\frac{3-4k^2}{7k^2+6} \right)^k$
Using the Root Test, let $a_k = \left(\frac{3-4k^2}{7k^2+6} \right)^k$.

$$\rho = \lim_{k\to\infty} \left| \left(\frac{3-4k^2}{7k^2+6} \right)^k \right|^{1/k} = \lim_{k\to\infty} \left| \frac{3-4k^2}{7k^2+6} \right| = \frac{4}{7} < 1$$

$\Rightarrow$ The series $\boxed{\text{converges absolutely}}$.

<u>Ex.</u>

$$\sum_{k=1}^{\infty} \frac{k^k}{5^{3k-1}}$$

Using the Root Test:

$$\rho = \lim_{k\to\infty} \left(\frac{k^k}{5^{3k-1}} \right)^{1/k} = \lim_{k\to\infty} \frac{\left(k^k\right)^{1/k}}{\left(5^{3k-1}\right)^{1/k}}$$

$$= \lim_{k\to\infty} \frac{k}{5^{3-1/k}} = \frac{\infty}{125} = \infty$$

$\Rightarrow$ So, the series $\boxed{\text{diverges}}$.